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Homogeneous geodesics in homogeneous Finsler spaces

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Abstract

In this paper, we study homogeneous geodesics in homogeneous Finsler spaces. We first give a simple criterion that characterizes geodesic vectors. We show that the geodesics on a Lie group, relative to a bi-invariant Finsler metric, are the cosets of the one-parameter subgroups. The existence of infinitely many homogeneous geodesics on the compact semi-simple Lie group is established. We introduce the notion of a naturally reductive homogeneous Finsler space. As a special case, we study homogeneous geodesics in homogeneous Randers spaces. Finally, we study some curvature properties of homogeneous geodesics. In particular, we prove that the **S**-curvature vanishes along the homogeneous geodesics.

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1. Introduction

A connected Riemannian manifold (M, g) is said to be *homogeneous* if a connected group of isometries G acts transitively on it. Such an M can be identified with $(\frac{G}{H}, g)$, where H is the isotropy group at a fixed point o of M. The Lie algebra $\underline{\mathbf{g}}$ of G admits a reductive decomposition $\underline{\mathbf{g}} = \mathbf{m} \oplus \underline{\mathbf{h}}$, where $\mathbf{m} \subset \underline{\mathbf{g}}$ is a subspace of $\underline{\mathbf{g}}$ isomorphic to the tangent space $T_o M$ and $\underline{\mathbf{h}}$ is the Lie algebra of H [6]. In general, such a decomposition is not unique. A *homogeneous geodesic* through the origin $o \in M = \frac{G}{H}$ is a geodesic $\gamma(t)$ which is an orbit of a one-parameter subgroup of G, that is

$$\gamma(t) = \exp(tZ)(o), \quad t \in R$$

where Z is a non-zero vector of $\underline{\mathbf{g}}$.

Homogeneous geodesics have important applications to mechanics. For example, the equation of motion of many systems of classical mechanics reduces to the geodesic equation in an appropriate Riemannian manifold M.

Geodesics of left-invariant Riemannian metrics on Lie groups were studied by Arnold, extending Euler's theory of rigid-body motion [1]. A major part of Arnold's paper is devoted to the study of homogeneous geodesics. Homogeneous geodesics are called by Arnold "relative equilibriums". The description of such relative equilibria

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is important for qualitative description of the behaviour of the corresponding mechanical system with symmetries. There is a big literature in mechanics devoted to the investigation of relative equilibria. Studying the set of homogeneous geodesics of a homogeneous Riemannian manifold $(\frac{G}{H}, g)$, the concept of the geodesic vector proved to be convenient [8]. A non-zero vector $X \in \underline{\mathbf{g}}$ is called a geodesic vector if the curve $\gamma(t) = \exp(tZ)(o)$ is a geodesic on $(\frac{G}{H}, g)$. The following lemma can be found in [8].

Lemma 1.1. *A vector $X \in \underline{\mathbf{g}} - \{0\}$ is a geodesic vector if and only if*

$$\langle [X, Y]_m, X_m \rangle = 0 \quad \forall Y \in \mathbf{m},$$

where $\langle , \rangle$ is the $\mathrm{Ad}(H)$-invariant scalar product on $\mathbf{m}$ induced by the Riemannian scalar product on T_oM and the subscript m indicates the projection into $\mathbf{m}$. The study of the set of homogeneous geodesics of a homogeneous Riemannian manifold is obviously reducible to the study of the set of its geodesic vectors.

A Finsler metric on a manifold is a family of Minkowski norms on tangent spaces. There are several notions of curvature in Finsler geometry. The flag curvature $\mathbf{K}$ is an analogue of the sectional curvature in Riemannian geometry. The Cartan torsion $\mathbf{C}$ is a primary quantity which characterizes Riemannian metrics among Finsler metrics. There is another quantity which also characterizes Riemannian metrics among Finsler metrics, that is the so-called distortion τ. The horizontal derivative of τ along geodesics is the so-called $\mathbf{S}$-curvature $\mathbf{S} = \tau_{;k} y^k$. While many works have been done on the general geometric properties of Finsler geometry, such as connections, geodesics and curvature, only very little attention has been paid to the group aspects of this interesting field. This may be mainly due to the Myers–Steenrod theorem in Riemannian geometry not being successfully generalized to the Finslerian case for a rather long period. A proof of this theorem for the Finslerian case was given in [13,3]. Namely they proved that the group of isometries of a Finsler space is a Lie transformation group of the underlying manifold. This result opens a door to using Lie group theory to study Finsler geometry [4,9].

The purpose of the present paper is to study homogeneous geodesics in homogeneous Finsler spaces. The definition of homogeneous geodesics is similar to that in the Riemannian case.

2. Preliminaries

2.1. Finsler spaces

In this section, we recall briefly some known facts about Finsler spaces. For details, see [2].

Let M be a n-dimensional C^∞ manifold and $TM = \bigcup_{x \in M} T_xM$ the tangent bundle. If the continuous function $F : TM \longrightarrow R_+$ satisfies the condition that it is C^∞ on $TM \setminus \{0\}$; $F(tu) = tF(u)$ for all $t \geq 0$ and $u \in TM$, i.e., F is positively homogeneous of degree one; and for any tangent vector $y \in T_xM \setminus \{0\}$, the following bilinear symmetric form $g_y : T_xM \times T_xM \longrightarrow R$ is positive definite:

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(x, y + su + tv)]|_{s=t=0},$$

then we say that (M, F) is a Finsler manifold.

Let

$$g_{ij}(x, y) = \left(\frac{1}{2} F^2\right)_{y^i y^j} (x, y).$$

By the homogeneity of F, we have

$$g_y(u, v) = g_{ij}(x, y) u^i v^j, \qquad F(x, y) = \sqrt{g_{ij}(x, y) y^i y^j}.$$

Let $\gamma : [0, r] \longrightarrow M$ be a piecewise C^∞ curve. Its integral length is defined as

$$L(\gamma) = \int_0^r F(\gamma(t), \dot{\gamma}(t)) \mathrm{d}t.$$

For $x_0, x_1 \in M$ denote by $\Gamma(x_0, x_1)$ the set of all piecewise C^∞ curves $\gamma : [0, r] \longrightarrow M$ such that $\gamma(0) = x_0$ and $\gamma(r) = x_1$. Define a map $d_F : M \times M \longrightarrow [0, \infty)$ by

$$d_F(x_0, x_1) = \inf_{\gamma \in \Gamma(x_0, x_1)} L(\gamma).$$

Of course we have $d_F(x_0, x_1) \geq 0$, where the equality holds if and only if $x_0 = x_1$; $d_F(x_0, x_2) \leq d_F(x_0, x_1) + d_F(x_1, x_2)$. In general, since F is only a positive homogeneous function, $d_F(x_0, x_1) \neq d_F(x_1, x_0)$, and therefore (M, d_F) is only a non-reversible metric space.

Let $\pi^* TM$ be the pull-back of the tangent bundle TM by $\pi : TM \setminus \{0\} \longrightarrow M$. Unlike the Levi-Civita connection in Riemannian geometry, there is no unique natural connection in the Finsler case. Among these connections on $\pi^* TM$, we choose the *Chern connection* whose coefficients are denoted by Γ^i_{jk} (see [2, p. 38]). This connection is almost g-compatible and has no torsion. Here $g(x, y) = g_{ij}(x, y) dx^i \otimes dx^j = (\frac{1}{2} F^2)_{y^i y^j} dx^i \otimes dx^j$ is the Riemannian metric on the pulled-back bundle $\pi^* TM$.

The Chern connection defines the covariant derivative $D_V U$ of a vector field $U \in \chi(M)$ in the direction $V \in T_p M$. Since, in general, the Chern connection coefficients Γ^i_{jk} in natural coordinates have a directional dependence, we must say explicitly that $D_V U$ is defined with a fixed reference vector. In particular, let $\sigma : [0, r] \longrightarrow M$ be a smooth curve with velocity field $T = T(t) = \dot{\sigma}(t)$. Suppose that U and W are vector fields defined along σ. We define $D_T U$ with *reference vector* W as

$$D_T U = \left[\frac{\mathrm{d}U^i}{\mathrm{d}t} + U^j T^k (\Gamma^i_{jk})_{(\sigma, W)} \right] \frac{\partial}{\partial x^i} |_{\sigma(t)}.$$

A curve $\sigma : [0, r] \longrightarrow M$, with velocity $T = \dot{\sigma}$ is a Finslerian *geodesic* if

$D_T \left[\frac{T}{F(T)} \right] = 0$, with reference vector T.

We assume that all our geodesics $\sigma(t)$ have been parameterized to have constant Finslerian speed. That is, the length $F(T)$ is constant. These geodesics are characterized by the equation

$D_T T = 0$, with reference vector T.

Since $T = \frac{\mathrm{d}\sigma^i}{\mathrm{d}t} \frac{\partial}{\partial x^i}$, this equation says that

$$\frac{\mathrm{d}^2 \sigma^i}{\mathrm{d}t^2} + \frac{\mathrm{d}\sigma^j}{\mathrm{d}t} \frac{\mathrm{d}\sigma^k}{\mathrm{d}t} (\Gamma^i_{jk})_{(\sigma, T)} = 0.$$

If U, V and W are vector fields along a curve σ, which has velocity $T = \dot{\sigma}$, we have the derivative rule

$$\frac{\mathrm{d}}{\mathrm{d}t} g_W(U, V) = g_W(D_T U, V) + g_W(U, D_T V)$$

whenever $D_T U$ and $D_T V$ have reference vector W and one of the following conditions holds:

(i) U or V is proportional to W, or
(ii) $W = T$ and σ is a geodesic.

2.2. Homogeneous Finsler spaces

Let (M, F) be a Finsler space, where F is positively homogeneous. As in the Riemannian case, we have two types of definition of isometry on (M, F), in terms of Finsler function in the tangent space and the induced non-reversible distance function on the base manifold M. The equivalence of these two definitions in the Finsler case is a result of Deng and Hou [3]. They also prove that the group of isometries of a Finsler space is a Lie transformation group of the underlying manifold which can be used to study homogeneous Finsler spaces.

Definition 2.1. A Finsler space (M, F) is called a homogeneous Finsler space if the group of isometries of (M, F), $I(M, F)$, acts transitively on M.

A Finsler manifold (M, F) is said to be forward geodesically complete if every geodesic $\gamma(t)$, $a \leq t < b$, parameterized to have constant Finslerian speed, can be extended to a geodesic defined on $a \leq t < \infty$.

Theorem 2.2 (*[9]*). *Every homogeneous Finsler space is forward complete.*

Theorem 2.3 (*[4]*). *Let G be a Lie group, H be a closed subgroup of G. Suppose there exists an invariant Finsler metric on $\frac{G}{H}$. Then there exists an invariant Riemannian metric on $\frac{G}{H}$.*

Let $M = \frac{G}{H}$ be a homogeneous space, where H is the isotropy subgroup at a point $o \in M$. If the linear isotropy representation $\lambda : H \longrightarrow GL(M_o)$, $h \longrightarrow h_{*o}$, is faithful, that is, injective, then G acts effectively on M. Let (M, F) be a connected homogeneous Finsler manifold. If G is any connected transitive group of isometries of M and H is the isotropy subgroup at a point, then M is naturally identified with the homogeneous manifold $\frac{G}{H}$. The Finsler metric F on M can be considered as a G-invariant Finsler metric on $\frac{G}{H}$. By Theorem 2.3, there exists a G-invariant Riemannian metric on $\frac{G}{H}$. So the linear isotropy representation is faithful and G acts effectively on $\frac{G}{H}$.

A homogeneous space $\frac{G}{H}$ is called reductive if there exists a vector space decomposition $\underline{\mathbf{g}} = \mathbf{m} \oplus \underline{\mathbf{h}}$ such that $\mathrm{Ad}(H)\mathbf{m} \subset \mathbf{m}$. In this case $\mathbf{m} \oplus \underline{\mathbf{h}}$ is called a reductive decomposition of $\underline{\mathbf{g}}$. It is well known [6,7] that each Riemannian homogeneous space is reductive. We now have the following.

Remark 2.4. Any homogeneous Finsler manifold $M = \frac{G}{H}$ is a reductive homogeneous space.

3. Homogeneous geodesics in homogeneous Finsler spaces

Let $(M = \frac{G}{H}, F)$ be a homogeneous Finsler space with a fixed origin p. Let $\underline{\mathbf{g}}$ and $\underline{\mathbf{h}}$ be the Lie algebra of G and H respectively and let

$$\underline{\mathbf{g}} = \mathbf{m} \oplus \underline{\mathbf{h}}$$

be a reductive decomposition of the Lie algebra $\underline{\mathbf{g}}$. From Remark 2.4, such a decomposition always exists.

For each $X \in \underline{\mathbf{g}}$ we obtain the corresponding fundamental vector field X^* on M by means of

$$X^*_q = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(\exp(tX)q) \quad \forall q \in M.$$

The canonical projection $\pi : G \longrightarrow \frac{G}{H}$ induces an isomorphism between the subspace $\mathbf{m}$ and the tangent space T_pM. Identifying $\underline{\mathbf{g}}$ with T_eG we get $d\pi(X) = X^*_p$ for each $X \in \underline{\mathbf{g}}$ and hence $d\pi(X_m) = X^*_p$.

Using this natural identification and the scalar product $g_{X^*_p}$ on T_pM we obtain a scalar product g_{X_m} on $\mathbf{m}$.

A vector $X \in \underline{\mathbf{g}} - \{o\}$ will be called a *geodesic vector* if the curve $\gamma(t) = \exp(tX)(p)$ is a constant speed geodesic of (M, F).

Let $(M = \frac{G}{H}, g)$ be a Riemannian homogeneous space, and $\underline{\mathbf{g}} = \mathbf{m} \oplus \underline{\mathbf{h}}$ be a reductive decomposition. Kowalski and Vanhecke [8] proved that $X \in \underline{\mathbf{g}}$ is a geodesic vector if and only if

$$g([X, Y]_m, X_m) = 0 \quad \forall Y \in \mathbf{m}.$$

In the Finslerian case we get the following theorem. We use some ideas from [8] in our proof.

Theorem 3.1. *A vector $X \in \underline{\mathbf{g}} - \{0\}$ is geodesic vector if and only if*

$$g_{X_m}(X_m, [X, Z]_m) = 0 \quad \forall Z \in \underline{\mathbf{g}}.$$

Proof. Let (M, F) be a Finsler space. For any vector fields T, V, W on M, we have [2]

$$Tg_W(V, W) = g_W(D_TV, W) + g_W(V, D_TW) \quad \text{with reference } W. \tag{1}$$

Similarly,

$$Vg_W(T, W) = g_W(D_VT, W) + g_W(T, D_VW), \tag{2}$$

$$Wg_W(V, W) = g_W(D_WV, W) + g_W(V, D_WW). \tag{3}$$

All covariant derivatives have W as reference vector.

Subtracting (2) from the summation of (1) and (3) we get

$$g_W(V, D_{W+T}W) + g_W(W-T, D_V W) = Tg_W(V,W) - Vg_W(T,W) + Wg_W(V,W) - g_W([T,V],W) - g_W([W,V],W),$$

where we have used the symmetry of the connection, i.e., $D_V W - D_W V = [V, W]$. Setting $T = W - V$ in the above equation, we obtain

$$2g_W(V, D_W W) = 2Wg_W(V,W) - Vg_W(W,W) - 2g_W([W,V],W). \tag{4}$$

Let $X, Z \in \underline{\mathbf{g}}$ be given and denote by X^* and Z^* the corresponding fundamental vector fields on M. From the above equation we get

$$2g_{X*}(D_{X^*}X^*, Z^*) = 2X^*g_{X*}(X^*, Z^*) - Z^*g_{X*}(X^*, X^*) + 2g_{X*}([Z^*, X^*], X^*). \tag{5}$$

Recall also the formulas

$$\mathrm{Ad}(\exp(tX))Y = Y + t[X,Y] + O(t^2), \quad X, Y \in \underline{\mathbf{g}}, \tag{6}$$

$$k \cdot \exp(tX) \cdot k^{-1} = \exp(t\mathrm{Ad}(k)X), \quad k \in G, X \in \underline{\mathbf{g}}. \tag{7}$$

Define for brevity $g_t = \exp(tX)$, $h_s = \exp(sZ)$. Using (7) and (6), we get first, for any $x \in M$,

$$\begin{aligned} Z^*_{g_t(x)} = \frac{\mathrm{d}}{\mathrm{d}s}|_0\, h_s g_t(x) &= (dg_t)\frac{\mathrm{d}}{\mathrm{d}s}|_0\, g_t^{-1}h_s g_t(x) \\ &= (dg_t)\frac{\mathrm{d}}{\mathrm{d}s}|_0 \exp(s\mathrm{Ad}(g_t^{-1})Z)(x) \\ &= (dg_t)[\mathrm{Ad}(g_t^{-1})Z]^*_x \\ &= (dg_t)[\mathrm{Ad}(\exp(-tX))Z]^*_x \\ &= (dg_t)[Z - t[X,Z] + O(t^2)]^*_x. \end{aligned}$$

Similarly, we get

$$X^*_{h_s(x)} = (dh_s)[X - s[Z,X] + O(s^2)]^*_x.$$

We shall also use the obvious relations

$$X^*_{g_t(x)} = (dg_t)X^*_x, \qquad Z^*_{h_s(x)} = (dh_s)Z^*_x.$$

Since g_t is an isometry, dg_t is a linear isometry between the spaces T_pM and $T_{g_t(p)}M$, $\forall p \in M$. Therefore for any vector fields V, W on M we have

$$g_{dg_t(X^*)}(dg_t(V), dg_t(W)) = g_{X*}(V,W). \tag{8}$$

Now, we calculate

$$\begin{aligned} X^*_x g_{X*}(X^*, Z^*) &= \frac{\mathrm{d}}{\mathrm{d}t}|_0\, g_{X*}(X^*_{g_t(x)}, Z^*_{g_t(x)}) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}|_0\, g_{X*}(dg_t(X^*_x), (dg_t)[Z - t[X,Z] + O(t^2)]^*_x) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}|_0\, g_{X*}(X^*_x, Z^*_x + t[X^*, Z^*]_x + O(t^2)) \\ &= g_{X*}(X^*, [X^*, Z^*])(x). \end{aligned}$$

Further,

$$\begin{aligned} Z^*_x g_{X*}(X^*, X^*) &= \frac{\mathrm{d}}{\mathrm{d}s}|_0\, g_{X*}(X^*_x + s[Z^*, X^*]_x + O(s^2), X^*_x + s[Z^*, X^*]_x + O(s^2)) \\ &= 2g_{X*}(X^*, [Z^*, X^*])(x). \end{aligned}$$

Substituting into (5), we get on M

$$g_{X*}(D_{X^*}X^*, Z^*) = g_{X*}(X^*, [X^*, Z^*]) = -g_{X*}(X^*, [X, Z]^*). \tag{9}$$

Now, suppose first that X is a geodesic vector, i.e. $\gamma(t) = \exp(tX)(p)$ is a geodesic of F with Finslerian constant speed. Then $D_{X^*_{g_t(p)}} X^*_{g_t(p)} = 0$, so in particular,

$$g_{X*}(X^*_p, [X, Z]^*_p) = 0.$$

Using the natural identification of $\mathbf{m}$ and T_pM we obtain

$$g_{X_m}(X_m, [X, Z]_m) = 0.$$

Letting Z be an arbitrary vector field on M, by (5) and (8) we have

$$\begin{aligned}
2g_{dg_t(X^*)}(dg_t(Z), D_{dg_t X^*}dg_t X^*) &= 2(dg_t X^*)g_{dg_t(X^*)}(dg_t Z, dg_t X^*) \\
&\quad - dg_t(Z)g_{dg_t(X^*)}(dg_t X^*, dg_t X^*) + 2g_{dg_t(X^*)}([dg_t Z, dg_t X^*], dg_t X^*) \\
&= 2X^* g_{X*}(Z, X^*) - Zg_{X*}(X^*, X^*) + 2g_{X*}([Z, X^*], X^*) \\
&= 2g_{X*}(Z, D_{X^*}X^*).
\end{aligned}$$

Consequently

$$g_{dg_t(X^*)}(dg_t(Z), D_{dg_t X^*}dg_t X^*) = g_{dg_t(X^*)}(dg_t(Z), dg_t(D_{X^*}X^*)).$$

Since Z is arbitrary and $g_{dg_t(X^*)}(\cdot, \cdot)$ is an inner product, we have

$$D_{dg_t X^*}dg_t X^* = dg_t(D_{X^*}X^*). \tag{10}$$

On the other hand, suppose that $g_{X^*_p}(X^*_p, [X, Z]^*_p) = 0$. Then

$$\begin{aligned}
g_{X^*_{g_t(p)}}(X^*, [X, Z]^*)(\exp(tX)(p)) &= g_{X^*_{g_t(p)}}(dg_t X^*_p, dg_t[X, Z]^*_p) \\
&= g_{X^*_p}(X^*_p, [X, Z]^*_p) = 0
\end{aligned}$$

for any $Z \in \underline{\mathbf{g}}$. Then (9) yields that

$$\begin{aligned}
g_{X^*}(D_{X^*}X^*, Z)(\exp(tX)(p)) &= g_{X^*}(dg_t(D_{X^*}X^*)_p, dg_t Z^*_p) \\
&= g_{X^*}((D_{dg_t X^*}dg_t X^*)_p, dg_t Z^*_p) \\
&= g_{X^*}(D_{X^*_{g_t(p)}} X^*_{g_t(p)}, dg_t Z^*_p) = 0.
\end{aligned}$$

Then this yields that $\exp(tX)(p)$ is a geodesic with constant speed. □

Corollary 3.2. *A vector $X \in \underline{\mathbf{g}} - \{0\}$ is a geodesic vector if and only if*

$$g_{X_m}(X_m, [X, Y]_m) = 0 \quad \textit{for all } Y \in \mathbf{m}. \tag{11}$$

Proof. Since F is G-invariant, we have

$$F(\mathrm{Ad}(h)W) = F(W) \quad \forall h \in H, W \in \mathbf{m}.$$

Therefore, $\forall y \neq 0, u, v \in \mathbf{m}, x \in \underline{\mathbf{h}}, t, r, s \in R$, we have

$$F^2(\mathrm{Ad}(\exp(tx)))(y + ru + sv) = F^2(y + ru + sv).$$

By definition,

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial r \partial s} F^2(y + ru + sv)|_{r=s=0}.$$

Thus

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial r \partial s} F^2(\mathrm{Ad}(\exp(tx))(y + ru + sv))|_{r=s=0}.$$

Now for $w \in \mathbf{m}$, from (6) we have

$$\mathrm{Ad}(\exp(tx))w = w + t[x, w] + O(t^2).$$

Therefore

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial r \partial s} F^2(y + ru + sv + t[x, y + ru + sv] + O(t^2))|_{r=s=0}.$$

Taking the derivative with respect to t at $t = 0$, we get

$$0 = g_y([x, u], v) + g_y(u, [x, v]) + 2C_y([x, y], u, v), \tag{12}$$

where C_y is the Cartan tensor of F at y. It follows from the homogeneity of F that $C_y(y, v, w) = 0$. So we have

$$g_y([x, y], y) = 0.$$

For any $Z \in \underline{\mathbf{g}}$, where $Z = Y + A$ with $Y \in \mathbf{m}$, $A \in \underline{\mathbf{h}}$, we obtain

$$g_{X_m}([X, Z]_m, X_m) = g_{X_m}([X, Y]_m, X_m) + g_{X_m}([X, A]_m, X_m).$$

Here, the second term is equal to $g_{X_m}([X_m, A], X_m) = 0$. Hence (11) implies that X is a geodesic vector. □

Corollary 3.3. *If $X \in \underline{\mathbf{g}} - \{0\}$ is a geodesic vector then $\mathrm{Ad}(h)X$ and λX are geodesic vectors for all $h \in H$, $\lambda \in R$.*

Proof. It is evident from the fact that

$$g_y(u, v) = g_{\mathrm{Ad}(h)y}(\mathrm{Ad}(h)u, \mathrm{Ad}(h)v) \quad \forall h \in H. \quad \square$$

In the following theorem, we consider bi-invariant Finsler metrics on Lie groups. We show that the geodesics of G starting at the identity element are the one-parameter subgroup of G. Let G be a connected Lie group. Deng and Hou [4] prove that there exists a bi-invariant Finsler metric on G if and only if there exists a Minkowski norm F on $\underline{\mathbf{g}}$ such that

$$g_y([x, u], v) + g_y(u, [x, v]) + 2C_y([x, y], u, v) = 0$$

$\forall y \in \underline{\mathbf{g}} - \{0\}$, $x, u, v \in \underline{\mathbf{g}}$. So we have the following:

Theorem 3.4. *Let G be a connected Lie group furnished with a bi-invariant Finsler metric F. Then each vector of $\underline{\mathbf{g}}$ is a geodesic vector.*

Here we study the existence of homogeneous geodesics in homogeneous Finsler spaces. The problem of the existence of homogeneous geodesics in homogeneous Finsler manifolds seems to be an interesting one. Concerning the existence of homogeneous geodesics in a general homogeneous Riemannian manifold, we have, at first, a result due to Kajzer who proved that a Lie group endowed with a left-invariant metric admits at least one homogeneous geodesic [5]. More recently Kowalski and Szenthe extended this result to all homogeneous Riemannian manifolds [7]. Homogeneous geodesics of left-invariant Lagrangian on Lie groups were studied by Szenthe [14]. The following result is due to Szenthe [14].

Theorem 3.5. *Let G be a compact connected Lie group and $L : TG \longrightarrow R$ a left-invariant Lagrangian which is a first integral of its Lagrangian field. Then L has at least one homogeneous geodesic. If, in particular, G is also semi-simple and of rank ≥ 2 then L has infinitely many homogeneous geodesics.*

Let (M, F) be a Finsler space. For every smooth parameterized curve $\gamma : [0, 1] \longrightarrow M$, the length of γ is given by

$$L(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t))\mathrm{d}t. \tag{13}$$

A geodesic of the Finsler space (M, F) is an extremal curve of (13). This is in fact a solution of the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial F}{\partial \dot{x}^i}\right) - \frac{\partial F}{\partial x^i} = 0, \quad \dot{x}^i = \frac{\mathrm{d}x^i}{\mathrm{d}t} \tag{14}$$

where $(x^i(t))$ is a local coordinate expression of γ. This system is equivalent to

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + 2G^i\left(x, \frac{\mathrm{d}x}{\mathrm{d}t}\right) = 0 \tag{15}$$

where

$$G^j(y) = \frac{1}{4} g^{jl}(y)\left[2\frac{\partial g_{sl}}{\partial x^k}(y) - \frac{\partial g_{sk}}{\partial x^l}(y)\right] y^s y^k.$$

Let

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}.$$

G is a vector field on $TM - \{0\}$. It is easy to see that $x(t)$ is a solution of (15) if and only if its lift $\dot{x}(t) = (x(t), \frac{\mathrm{d}x}{\mathrm{d}t}(t))$ is an integral curve of G in $TM - \{0\}$. G is called the geodesic spray. The following lemma shows that any Finsler metric F is a first integral of its geodesic spray.

Lemma 3.6 (*[11]*). *For any Finsler metric F on a manifold, $G(F) = 0$.*

So from Theorem 3.5 the following result follows:

Theorem 3.7. *Let G be a compact connected Lie group and F a left-invariant Finsler metric. Then F has at least one homogeneous geodesic. If, in particular, G is also semi-simple and of rank ≥ 2 then F has infinitely many homogeneous geodesics.*

3.1. Naturally reductive homogeneous Finsler space

The scheme is to treat the geometry of coset manifolds $\frac{G}{H}$ as a generalization of the geometry of Lie group G (since $\frac{G}{H}$ reduces to G when $H = \{e\}$). From this viewpoint, the isomorphism $\mathbf{m} \simeq T_o(\frac{G}{H})$ generalizes the canonical isomorphism $\underline{\mathbf{g}} \simeq T_e G$, and a G-invariant Riemannian metric on $\frac{G}{H}$ generalizes a left-invariant metric on G. The notion of the bi-invariant Riemannian metric on G generalizes as follows.

Definition 3.8. A Riemannian homogeneous space $(\frac{G}{H}, g)$ is said to be naturally reductive if there exists a reductive decomposition $\underline{\mathbf{g}} = \mathbf{m} + \underline{\mathbf{h}}$ of $\underline{\mathbf{g}}$ satisfying the condition

$$\langle [X, Y]_m, Z\rangle + \langle Y, [X, Z]_m\rangle = 0 \tag{16}$$

for all $X, Y, Z \in \mathbf{m}$.

where $\langle , \rangle$ denotes the inner product on $\mathbf{m}$ induced by the metric g.

In fact, when $H = \{e\}$, and hence $\mathbf{m} = \underline{\mathbf{g}}$, the above condition is just the condition

$$\langle [X, Y], Z\rangle + \langle Y, [X, Z]\rangle = 0, \tag{17}$$

for a bi-invariant Riemannian metric on G. In [4] the authors introduced the notion of a Minkowski Lie algebra:

Definition 3.9. Let $\underline{\mathbf{g}}$ be a real Lie algebra, F be a Minkowski norm on $\underline{\mathbf{g}}$. Then $\{\underline{\mathbf{g}}, F\}$ is called a Minkowski Lie algebra if the following condition is satisfied:

$$g_y([x, u], v) + g_y(u, [x, v]) + 2C_y([x, y], u, v) = 0 \tag{18}$$

where $y \in \underline{\mathbf{g}} - \{0\}$, $x, u, v \in \underline{\mathbf{g}}$.

The following was shown:

Theorem 3.10. *Let G be a connected Lie group. Then there exists a bi-invariant Finsler metric on G if and only if there exists a Minkowski norm F on $\underline{\mathbf{g}}$ such that $\{\underline{\mathbf{g}}, F\}$ is a Minkowski Lie algebra.*

It is easy to see that the notion of the Minkowski Lie algebra is the natural generalization of (17). Now we define the notion of a naturally reductive homogeneous Finsler space.

Definition 3.11. A homogeneous manifold $\frac{G}{H}$ with an invariant Finsler metric F is called naturally reductive if there exists an $\mathrm{Ad}(H)$-invariant decomposition $\underline{\mathbf{g}} = \underline{\mathbf{h}} + \mathbf{m}$ such that

$$g_y([x, u]_m, v) + g_y(u, [x, v]_m) + 2C_y([x, y]_m, u, v) = 0$$

where $y \neq 0, x, u, v \in \mathbf{m}$.

Evidently this definition is the natural generalization of (16). On the other hand, when $H = \{e\}$, and hence $\mathbf{m} = \underline{\mathbf{g}}$, this formula is just (18). The following theorem is a consequence of Theorem 3.1 and Definition 3.11.

Theorem 3.12. *Let $(\frac{G}{H}, F)$ be a naturally reductive homogeneous Finsler space. Then each geodesic of $(\frac{G}{H}, F)$ is an orbit of a one-parameter group of isometries $\{\exp(tX)\}$, $X \in \underline{\mathbf{g}}$.*

4. Homogeneous geodesics of Randers spaces

In this section, we consider homogeneous geodesics in a homogeneous Randers space. Randers metrics were introduced by Randers in 1941 [10] in the context of general relativity. They are Finsler spaces built from

(i) a Riemannian metric $\widetilde{a} = \widetilde{a}_{ij} dx^i \otimes dx^j$, and
(ii) a 1-form $\widetilde{b} = \widetilde{b}_i dx^i$,

both living globally on the smooth n-dimensional manifold M. The Finsler function of a Randers metric has the simple form $F = \alpha + \beta$, where

$$\alpha(x, y) = \sqrt{\widetilde{a}_{ij}(x) y^i y^j}, \qquad \beta(x, y) = \widetilde{b}_i(x) y^i.$$

Generic Randers metrics are only positively homogeneous. No Randers metric can satisfy absolute homogeneity $F(x, cy) = |c| F(x, y)$ unless $\widetilde{b} = 0$, in which case it is Riemannian. Also, in order for F to be positive and strongly convex on $TM \setminus \{0\}$, it is necessary and sufficient to have

$$\|\widetilde{b}\| = \sqrt{\widetilde{b}_i \widetilde{b^i}} < 1, \quad \text{where } \widetilde{b^i} = \widetilde{a}^{ij} \widetilde{b}_j.$$

See [2]. Strong convexity means that the fundamental tensor g_{ij} is positive definite. The Riemannian metric $\widetilde{a} = \widetilde{a}_{ij} dx^i \otimes dx^j$ induces the musical bijections between 1-forms and vector fields on M, namely $\flat : T_x M \longrightarrow T_x^* M$ given by $y \longrightarrow \widetilde{a}_x(y, \circ)$ and its inverse $\sharp : T_x^* M \longrightarrow T_x M$. In the local coordinates we have

$$(y^\flat)_i = \widetilde{a}_{ij} y^j \qquad (\theta^\sharp)^i = \widetilde{a}^{ij} \theta_j \quad y \in T_x M \ \theta \in T_x^* M.$$

Now the vector field corresponding to the 1-form $\widetilde{b}$ will be denoted by $\widetilde{b}^\sharp$; obviously we have $\|\widetilde{b}\| = \|\widetilde{b}^\sharp\|$ and

$$\beta(x, y) = (\widetilde{b}^\sharp)^\flat(y) = \widetilde{a}_x(\widetilde{b}^\sharp, y).$$

Thus a Randers metric F with Riemannian metric $\widetilde{a} = \widetilde{a}_{ij} dx^i \otimes dx^j$ and 1-form $\widetilde{b}$ can be shown by

$$F(x, y) = \sqrt{\widetilde{a}_x(y, y)} + \widetilde{a}_x(\widetilde{b}^\sharp, y) \quad x \in M, y \in T_x M,$$

where $\widetilde{a}_x(\widetilde{b}^\sharp, \widetilde{b}^\sharp) < 1, \forall x \in M$.

Theorem 4.1. *Let (M, F) be a homogeneous Randers space with F defined by the Riemannian metric $\widetilde{a}$ and the vector field X. Then X is a geodesic vector of $(M, \widetilde{a})$ if and only if X is a geodesic vector of (M, F).*

Proof. Let $F(p, y) = \sqrt{\widetilde{a}_p(y, y)} + \widetilde{a}_p(X, y)$.

Now for $s, t \in R$

$$F^2(y + su + tv) = \widetilde{a}(y + su + tv, y + su + tv) + \widetilde{a}^2(X, y + su + tv) + 2\sqrt{\widetilde{a}(y + su + tv, y + su + tv)}\widetilde{a}(X, y + su + tv).$$

By definition

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial r \partial s} F^2(y + ru + sv)|_{r=s=0}.$$

So by a direct computation we get

$$g_y(u, v) = \widetilde{a}(u, v) + \widetilde{a}(X, u)\widetilde{a}(X, v) + \frac{\widetilde{a}(u, v)\widetilde{a}(X, y)}{\sqrt{\widetilde{a}(y, y)}} - \frac{\widetilde{a}(v, y)\widetilde{a}(u, y)\widetilde{a}(X, y)}{\widetilde{a}(y, y)\sqrt{\widetilde{a}(y, y)}} + \frac{\widetilde{a}(X, v)\widetilde{a}(u, y)}{\sqrt{\widetilde{a}(y, y)}} + \frac{\widetilde{a}(X, u)\widetilde{a}(v, y)}{\sqrt{\widetilde{a}(y, y)}}. \tag{19}$$

So for all $Z \in \mathbf{m}$ we have

$$g_X(X, [X, Z]_m) = \widetilde{a}(X, [X, Z]_m) + \widetilde{a}(X, X)\widetilde{a}(X, [X, Z]_m) + 2\sqrt{\widetilde{a}(X, X)}\widetilde{a}(X, [X, Z]_m) = \widetilde{a}(X, [X, Z]_m)(1 + \sqrt{\widetilde{a}(X, X)} + F(X)).$$

Thus $g_X(X, [X, Z]_m) = 0$ if and only if $\widetilde{a}(X, [X, Z]_m) = 0$. □

Theorem 4.2. *Let (M, F) be a homogeneous Randers space with F defined by the Riemannian metric $\widetilde{a}$ and the vector field X. Let $y \in \underline{\mathbf{g}}$ be a vector for which $\widetilde{a}(X, [y, z]_m) = 0$ for all $z \in \mathbf{m}$. Then y is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \widetilde{a})$.*

Proof. According to the above formula for $g_y(u, v)$, we have

$$g_{y_m}(y_m, [y, z]_m) = \widetilde{a}(y_m, [y, z]_m) + \widetilde{a}(X, y_m)\widetilde{a}(X, [y, z]_m) + \frac{\widetilde{a}(y_m, [y, z]_m)\widetilde{a}(X, y_m)}{\sqrt{\widetilde{a}(y_m, y_m)}} + \widetilde{a}(X, [y, z]_m)\sqrt{\widetilde{a}(y_m, y_m)}$$
$$= \widetilde{a}(y_m, [y, z]_m)\left(1 + \frac{\widetilde{a}(X, y_m)}{\sqrt{\widetilde{a}(y_m, y_m)}}\right) + \widetilde{a}(X, [y, z]_m)\left(\widetilde{a}(X, y_m) + \sqrt{\widetilde{a}(y_m, y_m)}\right).$$

So we have

$$g_{y_m}(y_m, [y, z]_m) = \widetilde{a}(y_m, [y, z]_m)\left(\frac{F(y_m)}{\sqrt{\widetilde{a}(y_m, y_m)}}\right) + \widetilde{a}(X, [y, z]_m)F(y_m).$$

This concludes the proof. □

Let (M, F) be a Finsler space. Then (M, F) is called a Berwald space if the Chern connection coefficients $\Gamma^k_{ij}(x, y)$ in natural coordinate systems have no dependence on the vector y, or in other words, if the Chern connection defines a linear connection directly on the underlying manifold.

Theorem 4.3. *Let (M, F) be a homogeneous Randers space with F defined by the Riemannian metric $\widetilde{a} = \widetilde{a}_{ij} dx^i \otimes dx^j$ and the vector field X which is of Berwald type. Then (M, F) is naturally reductive if and only if the underlying Riemannian metric $(M, \widetilde{a})$ is naturally reductive.*

Proof. Let $(M, \widetilde{a})$ be naturally reductive. We show that for all $0 \neq y, z, u, v \in \mathbf{m}$

$$g_y([z, u]_m, v) + g_y(u, [z, v]_m) + 2C_y([z, y]_m, u, v) = 0.$$

Since F is of Berwald type, (M, F) and $(M, \widetilde{a})$ have the same connection. So according to the relation

$$g_{y_m}(y_m, [y, z]_m) = \widetilde{a}(y_m, [y, z]_m)\left(\frac{F(y_m)}{\sqrt{\widetilde{a}(y_m, y_m)}}\right) + \widetilde{a}(X, [y, z]_m)F(y_m),$$

for all $0 \neq y \in \mathbf{m}$ we have

$$\widetilde{a}(X, [y, z]_m) = 0 \quad \forall z \in \mathbf{m}.$$

From (19) we get

$$g_y([z, u]_m, v) = \widetilde{a}([z, u]_m, v)\left(1 + \frac{\widetilde{a}(X, y)}{\sqrt{\widetilde{a}(y, y)}}\right) + \widetilde{a}([z, u]_m, y)\left(\frac{\widetilde{a}(X, v)}{\sqrt{\widetilde{a}(y, y)}} - \frac{\widetilde{a}(v, y)\widetilde{a}(X, y)}{\widetilde{a}(y, y)\sqrt{\widetilde{a}(y, y)}}\right)$$
$$g_y(u, [z, v]_m) = \widetilde{a}(u, [z, v]_m)\left(1 + \frac{\widetilde{a}(X, y)}{\sqrt{\widetilde{a}(y, y)}}\right) + \widetilde{a}([z, v]_m, y)\left(\frac{\widetilde{a}(X, u)}{\sqrt{\widetilde{a}(y, y)}} - \frac{\widetilde{a}(u, y)\widetilde{a}(X, y)}{\widetilde{a}(y, y)\sqrt{\widetilde{a}(y, y)}}\right).$$

By definition

$$C_y(z, u, v) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\widetilde{a}_{y+tv}(z, u)]|_{t=0}.$$

So by a direct computation we get

$$\begin{aligned}2C_y(z, u, v) = {} & \frac{\widetilde{a}(z, u)\widetilde{a}(X, v)\widetilde{a}(y, y) - \widetilde{a}(y, v)\widetilde{a}(z, u)\widetilde{a}(X, y)}{\widetilde{a}(y, y)\sqrt{\widetilde{a}(y, y)}} \\ & + \frac{\widetilde{a}(X, u)\widetilde{a}(z, v)\widetilde{a}(y, y) - \widetilde{a}(y, v)\widetilde{a}(X, u)\widetilde{a}(z, y)}{\widetilde{a}(y, y)\sqrt{\widetilde{a}(y, y)}} \\ & + \frac{\widetilde{a}(X, z)\widetilde{a}(u, v)\widetilde{a}(y, y) - \widetilde{a}(y, v)\widetilde{a}(X, z)\widetilde{a}(u, z)}{\widetilde{a}(y, y)\sqrt{\widetilde{a}(y, y)}} \\ & - \frac{\widetilde{a}(x, y)\widetilde{a}(u, y)\widetilde{a}(z, v) + \widetilde{a}(x, y)\widetilde{a}(u, v)\widetilde{a}(z, y) + \widetilde{a}(x, v)\widetilde{a}(u, y)\widetilde{a}(z, y)}{\widetilde{a}(y, y)\sqrt{\widetilde{a}(y, y)}} \\ & + \frac{3\widetilde{a}(y, v)\widetilde{a}(X, y)\widetilde{a}(u, y)\widetilde{a}(z, y)}{\widetilde{a}(y, y)^2\sqrt{\widetilde{a}(y, y)}}.\end{aligned}$$

So we have

$$\begin{aligned}C_y([z, y]_m, u, v) = {} & \widetilde{a}([z, y]_m, u)\left(\frac{\widetilde{a}(X, v)}{\sqrt{\widetilde{a}(y, y)}} - \frac{\widetilde{a}(y, v)\widetilde{a}(X, y)}{\widetilde{a}(y, y)\sqrt{\widetilde{a}(y, y)}}\right) \\ & + \widetilde{a}([z, y]_m, v)\left(\frac{\widetilde{a}(X, u)}{\sqrt{\widetilde{a}(y, y)}} - \frac{\widetilde{a}(X, y)\widetilde{a}(u, y)}{\widetilde{a}(y, y)\sqrt{\widetilde{a}(y, y)}}\right).\end{aligned}$$

Therefore

$$\begin{aligned}& g_y([z, u]_m, v) + g_y(u, [z, v]_m) + 2C_y([z, y]_m, u, v) = (\widetilde{a}([z, u]_m, v) + \widetilde{a}(u, [z, v]_m))\left(1 + \frac{\widetilde{a}(X, y)}{\sqrt{\widetilde{a}(y, y)}}\right) \\ & \quad + (\widetilde{a}([z, u]_m, y) + \widetilde{a}([z, y]_m, u))\left(\frac{\widetilde{a}(X, v)}{\widetilde{a}(y, y)} - \frac{\widetilde{a}(v, y)\widetilde{a}(X, y)}{\widetilde{a}(y, y)\sqrt{\widetilde{a}(y, y)}}\right) \\ & \quad + (\widetilde{a}([z, v]_m, y) + \widetilde{a}([z, y]_m, v))\left(\frac{\widetilde{a}(X, u)}{\sqrt{\widetilde{a}(y, y)}} - \frac{\widetilde{a}(X, y)\widetilde{a}(u, y)}{\widetilde{a}(y, y)\sqrt{\widetilde{a}(y, y)}}\right).\end{aligned}$$

Thus

$$g_y([z, u]_m, v) + g_y(u, [z, v]_m) + 2C_y([z, y]_m, u, v) = 0$$

for all $y \neq 0, u, v, z \in \mathbf{m}$.

Conversely let (M, F) be naturally reductive, i.e. for all $y \neq 0, u, v, z \in \mathbf{m}$,

$$g_y([z, u]_m, v) + g_y(u, [z, v]_m) + 2C_y([z, y]_m, u, v) = 0.$$

So

$$g_y([y, u]_m, v) + g_y(u, [y, v]_m) = 0.$$

(M, F) and $(M, \widetilde{a})$ have the same geodesics and for all $0 \neq y \in \mathbf{m}$, y is a geodesic vector, so for all $y \in \mathbf{m}$ we have

$$\widetilde{a}(X, [y, z]_m) = 0 \quad \forall z \in \mathbf{m}.$$

Therefore

$$g_y([y,u]_m, v) = \widetilde{a}([y,u]_m, v)\left(1 + \frac{\widetilde{a}(X,y)}{\sqrt{\widetilde{a}(y,y)}}\right),$$

$$g_y(u, [y,v]_m) = \widetilde{a}(u, [y,v]_m)\left(1 + \frac{\widetilde{a}(X,y)}{\sqrt{\widetilde{a}(y,y)}}\right).$$

So we have

$$g_y([y,u]_m, v) + g_y(u,[y,v]_m) = (\widetilde{a}([y,u]_m, v) + \widetilde{a}(u,[y,v]_m))\left(1 + \frac{\widetilde{a}(X,y)}{\sqrt{\widetilde{a}(y,y)}}\right) = 0.$$

We easily see that $\left(1 + \frac{\widetilde{a}(X,y)}{\sqrt{\widetilde{a}(y,y)}}\right) \neq 0$. Thus

$$\widetilde{a}([y,u]_m, v) + \widetilde{a}(u,[y,v]_m) = 0. \quad \square$$

5. Some curvature properties

The **S**-curvature is one of most important non-Riemannian quantities in Finsler geometry which vanishes for Riemannian metrics. The **S**-curvature has been introduced in [12]. In this section, we discuss the relationship between the homogeneous geodesics and **S**-curvature.

Let F be a Finsler metric on a manifold M. Let $\{e_i\}_{i=1}^n$ be a basis for T_xM and $\{\omega_i\}_{i=1}^n$ the dual basis for T_x^*M. Denote by $d\mu_x = \sigma(x)\omega^1 \wedge \cdots \wedge \omega^n$ the Busemann volume form at x, where

$$\sigma(x) = \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\{(y^i) \in R^n, F(y^i e_i) < 1\}},$$

where B^n denotes the unit ball in R^n and Vol denotes the Euclidean measure on R^n. For each $y \in T_xM - \{0\}$, define

$$\tau(x,y) = Ln\left[\frac{\sqrt{\det g_y(e_i, e_j)}}{\sigma(x)}\right].$$

The scalar function $\tau : TM \setminus \{0\} \longrightarrow R$ is called the *distortion*. To measure the rates of change of the distortion along geodesics, we define

$$\mathbf{S}(x,y) = \frac{\mathrm{d}}{\mathrm{d}t}[\tau(c(t), \dot{c}(t))]_{t=0}$$

where $c(t)$ is the geodesic with $\dot{c}(0) = y$.

The scalar function $\mathbf{S} : TM \setminus \{0\} \longrightarrow R$ is called the **S**-curvature [11,12].

Theorem 5.1. *Let X be a geodesic vector. Then $\mathbf{S}(X_m) = 0$ for the Busemann volume form.*

Proof. Let $\gamma(t) = \exp(tX)(p)$ be the homogeneous geodesic corresponding to X. Define $g_t = \exp(tX)$; obviously we have

$$X^*_{g_t(x)} = (dg_t)(X^*_x).$$

Take an arbitrary basis $\{e_i\}_{i=1}^n$ for T_xM. We obtain a frame along $\gamma(t)$,

$$e_i(t) = (dg_t)e_i \quad i = 1, \ldots, n.$$

Let $d\mu$ denote the Busemann volume form of F. Put

$$d\mu|_{c(t)} = \sigma(t)\omega^1(t) \wedge \cdots \wedge \omega^n(t),$$

where $\{\omega^i(t)\}$ is the basis for $T^*_{\gamma(t)}M$ which is dual to $\{e_i(t)\}_{i=1}^n$.

$$\sigma(t) = \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\{(y^i) \in R^n, F(y^i e_i(t)) < 1\}}.$$

Since g_t is an isometry we have $F(y^i e_i(t)) = F(y^i e_i)$ and by the relation (8) we know that

$$g_{\dot{\gamma}(t)}(e_i(t), e_j(t)) = g_{\dot{\gamma}(0)}(e_i, e_j).$$

Thus

$$\det\left[g_{\dot{\gamma}(t)}(e_i(t), e_j(t))\right] = \det\left[g_{\dot{\gamma}(0)}(e_i, e_j)\right],$$
$$\{(y^i) \in R^n, F(y^i e_i(t)) < 1\} = \{(y^i) \in R^n, F(y^i e_i) < 1\}.$$

This implies that

$$\tau(\gamma(t), \dot{\gamma}(t)) = Ln\left[\frac{\sqrt{\det[g_{\dot{\gamma}(t)}(e_i(t), e_j(t))]}}{\sigma(t)}\right]$$

is constant, so $\mathbf{S}(\gamma(t), \dot{\gamma}(t)) = 0$. In particular, at $t = 0$, $\mathbf{S}(X_p^*) = 0$. □

Now like for Riemannian spaces we define the notion of the *geodesic orbit (g.o.)* space for Finsler spaces.

Definition 5.2. A homogeneous Finsler space (M, F) is said to be geodesic orbit (g.o.) space if every geodesic in M is an orbit of a one-parameter group of isometries, i.e. there exists a transitive group G of isometries such that every geodesic in M is of the form $\exp(tX)p$ with $X \in \underline{\mathbf{g}}$, $p \in M$.

Corollary 5.3. *Let (M, F) be a g.o. Finsler space. Then the* $\mathbf{S}$*-curvature* $\mathbf{S} = 0$ *for the Busemann volume form.*

Proof. Let $0 \neq X \in T_x M$ be an arbitrary vector. Let γ be the geodesic with $\dot{\gamma}(0) = X$. Since (M, F) is a g.o. we can write

$$\gamma(t) = \exp(tX)\gamma(0).$$

According to Theorem 5.1, $\mathbf{S}(\gamma(t), \dot{\gamma}(t)) = 0$ and so $\mathbf{S}(x, X) = 0$. □

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